The Thrust Optimised Parabolic nozzle

Introduction

In the early 1950's, rocketeers attempted to devise 'the perfect nozzle': one that would cause the least thrust losses. They used the semi-algebraic Method of Characteristics (of supersonic flow) to devise such a nozzle, for whatever nozzle expansion ratio ϵ was required.

Unfortunately, at large expansion ratios, this nozzle was far too long, resulting in excess nozzle mass.

Rao in America and Shmyglevsky in Russia found a way to modify this method in order to produce an optimum nozzle that was much shorter: it resembled a church bell and was hence known as a 'bell nozzle'.

For ease of use, Rao measured the lengths of his resulting nozzles as fractions of the length of a standard 15 degree half-angle conical nozzle which has length:

$$L_{N\_cone} = \frac{(\sqrt{\epsilon}-1)\, R_t}{\tan(15)}$$ **equ. 1**   where R_t is the radius of the throat.

So an '80% bell' would have a length of 0.8 times this length.

Rao was clearly also schooled in traditional geometry. He was able to spot that the bell part of his bell nozzles could be approximated very closely by a skewed parabola allowing us to quickly sketch his nozzles with negligible loss of thrust performance.

These sketched approximations are known as Thrust Optimised Parabolic (TOP) nozzles, and have found use on a variety of actual launch vehicles because they perform better when over-expanded at ground-level altitude than the actual optimised bell nozzle (flow separation from the TOP nozzle wall is delayed at high back-pressure).

Rao's parabolas are known in Europe as quadratic Bézier curves, after French car bodywork draughtsman Pierre Bézier. Bézier curves are now used extensively in computer graphics.

The shape of the bell nozzle changes only minutely with the propellants used (varying ratio of specific heats γ) so one TOP nozzle methodology fits all propellants and is described below.

The geometric approach

Construction

The 'Rao nozzle' starts with Rao's preferred throat geometry, presumably a Rockedyne (where he worked) optimisation.
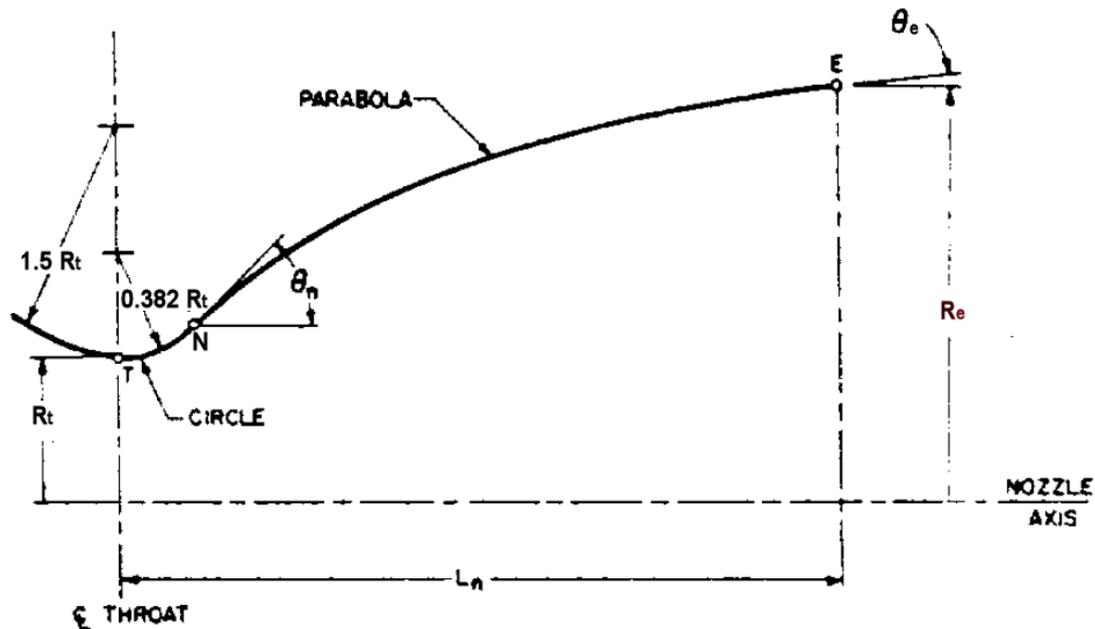
Two circular arcs are drawn:

The first curve, of radius 1.5 R_t, is drawn from an angle of say, -135 degrees, to the throat at -90 degrees (angles measured from the arc's origin). Then the second curve of radius 0.382 R_t is drawn from this angle of -90 degrees to an angle of (θ_n- 90) at inflection point N.
(R_t is the throat radius, R_e is the exit radius.)

Then a skewed parabola is drawn from point N to nozzle exit point E, tangent to the throat curve, and starting at an angle of θ_n and ending at an angle of θ_e.

The radius of the nozzle exit: $R_e = \sqrt{\epsilon}\, R_t$ **equ. 2**

and nozzle length $L_N = 0.8 \left(\dfrac{(\sqrt{\epsilon}-1)\, R_t}{\tan(15)} \right)$ **equ. 3**   for an 80% bell from equ. 1

Angles θ_n and θ_e were pre-calculated by Roe to match his bell nozzle, and presented as graphical data from which the following chart is reproduced for various percent lengths (from Ref. 1):
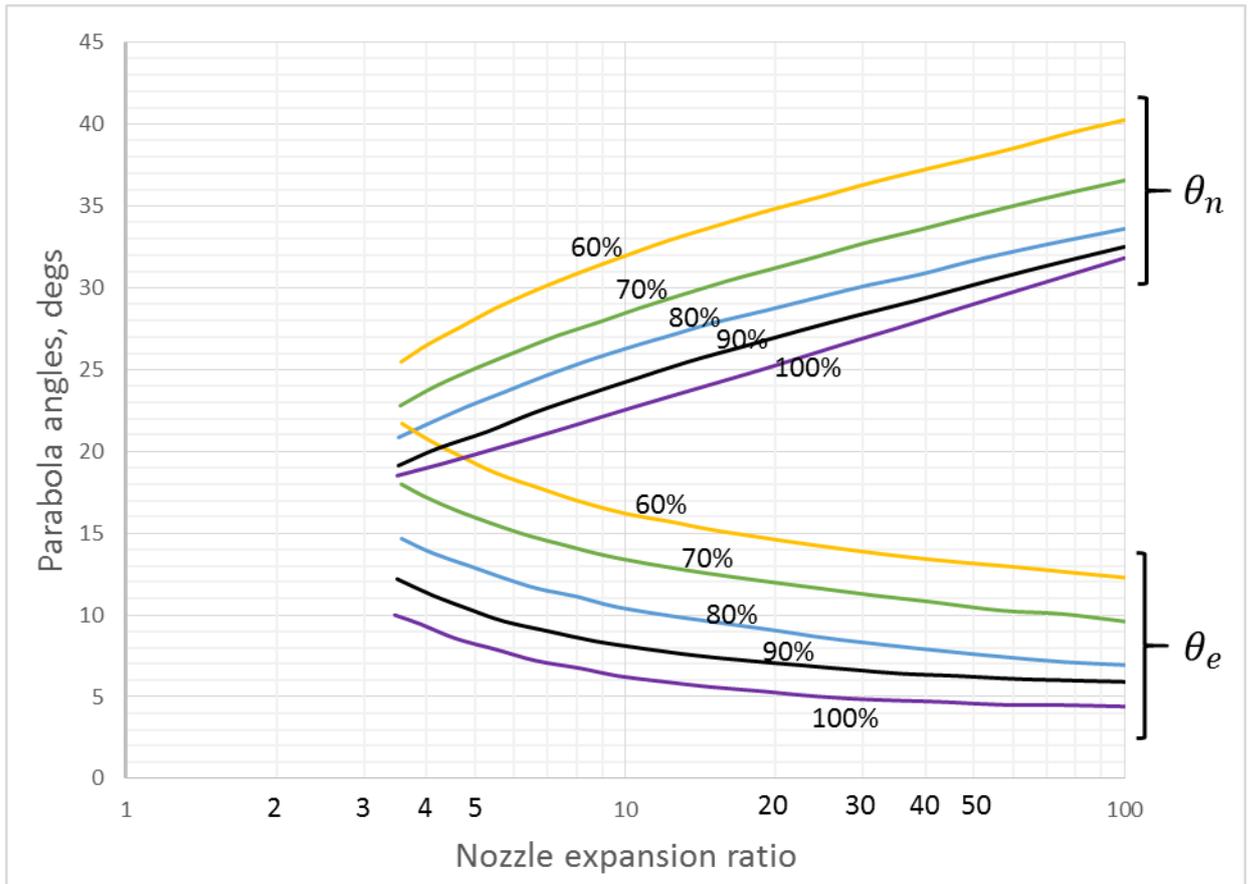
(The data for expansion ratios greater than 50 is extrapolated).

A parabola is then sketched-out using an ancient geometrical method for drawing a parabola detailed in Ref. 2.
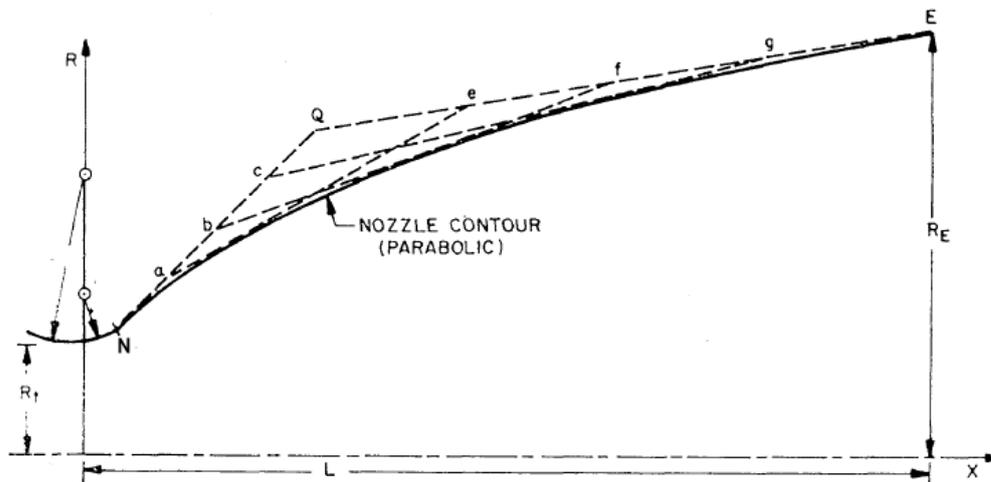
Straight lines are drawn at angles θ_n from point N, and θ_e back from point E, terminating where these lines cross at point Q.

Next, both of these lines are divided into an equal number of divisions (four in this example) These are labelled a,b,c and e,f,g.

A straight line is then drawn from point a to point e, then from b to f, and c to g. These form a mesh, the edge of which gives the parabola outline. The parabola is also tangent to the lines QN and QE.

Using many more divisions, on a CAD package for example, gives a sharper contour. Removing most of the mesh gives a series of straight-line segments: joining the midpoint of each segment with a smooth curve such as a CAD spline gives the nozzle contour.
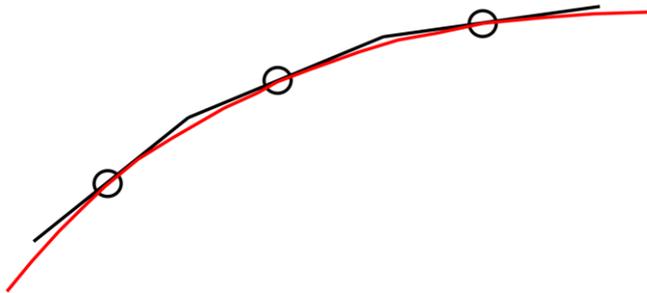
Alternately, interpolating along the construction lines gives the same points. In our example with four divisions and (4-1) construction lines, the parabola is defined at 1/4 the distance along line a-e, 2/4 along line b-f, and 3/4 along line c-f.

Efficiency
At a length ratio of 85% bell, a nozzle efficiency of 99% is reached, and only 0.2% of additional performance can be gained by increasing the length ratio to 100%. For this reason, 85% is often taken as upper bound. At length ratios below 70%, nozzle efficiency suffers. For these reasons, the 80% bell parabola is often chosen.

Example
An 80% bell nozzle with an area ratio of 70 (a typical upper stage nozzle) has angles $\theta_n = 33^o$ and $\theta_e = 7^o$ from the above chart. Drawn on a CAD package it looks like this (mesh removed for clarity):
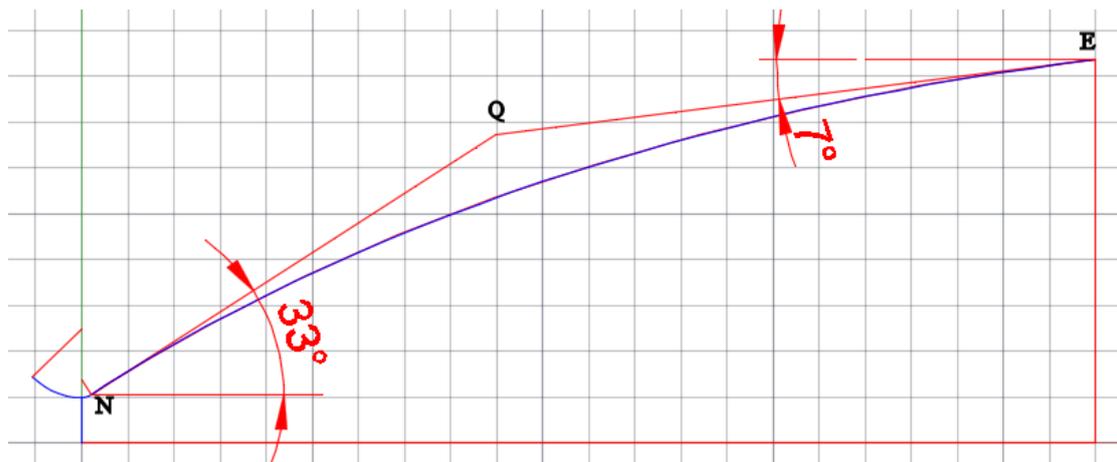
The mathematical approach

The throat
The equations of the above circular arcs defining the throat are defined trigonometrically, defining the origin of the coordinates as the centre of the narrowest part of the throat:

For the entrant section:

$x = 1.5\,R_t \cos\theta$
$y = 1.5\,R_t \sin\theta + 1.5\,R_t + R_t$ **equ.s 4**

where: $-135 \le \theta \le -90$

(The initial angle isn't defined and is up to the combustion chamber designer, -135 degrees is typical.)

For the exit section:

$x = 0.382\,R_t \cos\theta$
$y = 0.382\,R_t \sin\theta + 0.382\,R_t + R_t$ **equ.s 5**

where: $-90 \le \theta \le (\theta_n - 90)$

Throat contour
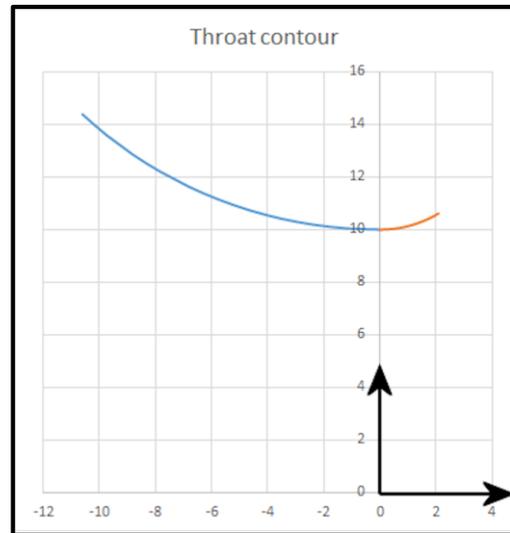
The bell
The bell is a quadratic Bézier curve, which has equations (see Wikipedia):

$x(t) = (1-t)^2 N_x + 2(1-t)t\,Q_x + t^2 E_x \qquad 0 \le t \le 1$
$y(t) = (1-t)^2 N_y + 2(1-t)t\,Q_y + t^2 E_y \qquad 0 \le t \le 1$ **equ.s 6**

Selecting equally spaced divisions between 0 and 1 produces the points described earlier in the graphical method, for example 0.25, 0.5, and 0.75.

Equations 6 are defined by points N, Q, and E (see the graphical method earlier for the locations of these points).

Point N is defined by equations 5 setting the angle to $(\theta n - 90)$.

Coordinate E_x is defined by equation 3, and coordinate E_y is defined by equation 2.

Point Q is the intersection of the lines: $\overrightarrow{NQ} = m_1 x + C_1$ and: $\overrightarrow{QE} = m_2 x + C_2$ **equ.s 7**

where: gradient $m_1 = \tan(\theta_n)$, gradient $m_2 = \tan(\theta_e)$ **equ.s 8**

and: intercept $C_1 = N_y - m_1 N_x$, intercept $C_2 = E_y - m_2 E_x$ **equ.s 9**

The intersection of these two lines (at point Q) is given by:

$$Q_x = \frac{(C_2 - C_1)}{(m_1 - m_2)}, \; Q_y = \frac{(m_1 C_2 - m_2 C_1)}{(m_1 - m_2)} \quad \textbf{equ.s 10}$$

References

1. "Liquid Rocket Engine Nozzles", NASA SP-8120, July 1976
2. "Exhaust Nozzle Contour for Optimum Thrust," G.V.R. Rao, Jet Propulsion 28, 377 (1958)
3. www.wikihow.com/Draw-a-Parabolic-Curve-(a-Curve-with-Straight-Lines)
4. Wikipedia: quadratic Bezier curve
5. www.waters.to/blog/rocket-nozzles-part-1-the-math/
6. Wikipedia: line-line intersection
7. www.wikihow.com/Find-the-Equation-of-a-Line